Exercise 1

Evaluate the line integral, where C is the given curve.

$$\int_C y\,ds, \quad C:\ x = t^2,\ y = 2t,\ 0 \le t \le 3$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}\int_C y\,ds &= \int_0^3 y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_0^3 (2t)\sqrt{(2t)^2 + (2)^2}\,dt \\ &= \int_0^3 2t\sqrt{4t^2+4}\,dt \\ &= 4\int_0^3 t\sqrt{t^2+1}\,dt.\end{aligned}$$

Make the following substitution.

$$\begin{aligned}u &= t^2 + 1 \\ du &= 2t\,dt \quad \rightarrow \quad \frac{du}{2} = t\,dt\end{aligned}$$

Therefore,

$$\begin{aligned}\int_C y\,ds &= 4\int_{0^2+1}^{3^2+1} \sqrt{u}\left(\frac{du}{2}\right) \\ &= 2\int_1^{10} u^{1/2}\,du \\ &= 2\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{10} \\ &= \frac{4}{3}\left(10^{3/2} - 1^{3/2}\right) \\ &= \frac{4}{3}\left(10\sqrt{10} - 1\right).\end{aligned}$$